Mathematics - 3 (Hons.)

Answer any six questions.

1. (a) State $\in - \delta$ definition of the limit of a function of one real variable and use the definition to show that $\lim_{x \to 1} f(x) = 5$, where $f(x) = 2x + 3$.

   (b) If a function $f(x)$ is continuous in a closed and bounded interval $[a, b]$, prove that the function $f(x)$ has a greatest and a least values.

2. (a) If $y' + y^{-1} = 2x$, prove that :

$$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

   (b) State and prove Euler's theorem for a homogeneous function of three independent variables.

3. (a) Find the condition that the straight line $x \cos \alpha + y \sin \alpha = p$ may touch the curve $x^m y^n = a^{m+n}$

   (b) Establish the formula $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ for a plane curve, where symbols have their usual meanings.

4. (a) For the curve $r = f(\theta)$, the radius of curvature ρ at any point is given by

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2} \text{ where } r_1 = \frac{dr}{d\theta} \text{ and } r_2 = \frac{d^2 r}{d^2\theta}.$$

   (b) Find the maximum value of $x^3 y^2 (1 - x - y)$, where $x \neq 0$, $y \neq 0$ and $x + y \neq 1$.

5. Evaluate any two of the following integrals :

   (a) $\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}}$ (b) $\int \frac{dx}{(x-\alpha)\sqrt{(\beta - x)}}$

   (c) $\int \cos^2 x \sin 4x \, dx$ (d) $\int e^{-x} \cos^2 x \, dx$

6. Evaluate any two of the following :

   (a) $\int_0^1 \tan^{-1} x \, dx$ (b) $\int_0^{\pi/2} \cos^n x \cos nx \, dx$

   (c) $\int_0^\infty \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2} dx$ (d) $\int_0^\pi \frac{dx}{5 + 4\cos x}$

7. (a) For any curve prove that, $s=\int \frac{r\,dr}{\sqrt{r^2-p^2}}$, where S denotes the arc length of the curve and other symbols have usual meanings.

(b) Find the whole area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

8. Find the surface area of a right cone whose semi-certical angle is a,, height h and base a circle of radius a. Also, find the volume of the cone.

9. (a) Define uniform continuity of a function of a real variable. Prove that a function continuous on a closed and bounded interval is uniformly continuous on it.

(b) If a function f is differentiable in [a, b] and $f(a)\neq f(b)$ then prove that $f(x)$ assumes all values between f (a) and f (b) at least once in [a, b].

10. (a) State and prove Cauchy's general principle of convergence of a sequence.

(b) If $\lim\limits_{x\to\infty} x_n = l$, prove that $\lim\limits_{x\to\infty} \frac{x_1+x_2+....x_n}{n}=l$.

11. (a) State and prove Leibnitz's rule regarding convergence of an alternating series.

(b) Test the convergence of the series $\sum\limits_{n=2}^{\infty}\frac{1}{(\log^n)^n}$.

12. (a) Define absolute convergence and conditional convergence of an infinite series of real numbers. Prove that every absolutely convergent series is convergent but not conversely.

(b) Test the convergence of any two of the following series :

(i) $\sum\limits_{n=1}^{\infty}\frac{\sqrt{(n+1)}-\sqrt{(n-1)}}{n}$.

(ii) $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.r^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+....to\,\infty$ (iii) $\sum\limits_{n=2}^{\infty}(\log n)^{\log n}$

(iv) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+...+\frac{(n-1)^{n-1}}{n^n}+....to\,\infty$